

3-MANIFOLDS MODULO SURGERY TRIANGLES

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1. INTRODUCTION

Floer homology theories can be used to define a number of different invariants of closed, oriented 3-manifolds. Many of these theories satisfy a “surgery triangle”, which is a relationship between the invariants of different Dehn surgeries $Y_\alpha(K)$ on a knot K in a fixed manifold $Y = Y_\infty(K)$. In general, if α, β, γ are three oriented surgery curves on the boundary torus of $Y \setminus \nu(K)$ such that

$$\alpha \cdot \beta = \beta \cdot \gamma = \gamma \cdot \alpha = -1$$

in integral homology, then there is an exact triangle of Floer homology groups,

$$\cdots \rightarrow HF(Y_\alpha(K)) \rightarrow HF(Y_\beta(K)) \rightarrow HF(Y_\gamma(K)) \rightarrow HF(Y_\alpha(K)) \rightarrow \cdots.$$

Surgery triangles are useful because one can always use them to express the invariants of a given manifold in terms of the invariants of simpler manifolds.

To make this observation precise, it is useful to borrow some language from finite geometry. We view diffeomorphism classes of oriented 3-manifolds as points in a geometry X , whose lines are surgery triangles $(Y_\alpha, Y_\beta, Y_\gamma)$. Note that some lines in this geometry may contain “doubled” points, because a manifold can be involved in a surgery triangle with itself (for example, consider Dehn surgeries on the unknot). We say that a set of 3-manifolds S is a subspace if each line meeting S in two (not necessarily distinct) points is entirely contained in S . We define the span of a set of 3-manifolds to be the smallest subspace that contains it. We say that a set of 3-manifolds is a generating set if its span is all of X . Using this terminology, we can precisely formulate the sense in which surgery triangles allow one to understand an arbitrary 3-manifold in terms of simpler manifolds.

Proposition 1. *The 3-sphere generates all closed 3-manifolds.*

Similar concepts apply to 3-manifolds with boundary.

Definition 1. *Let Σ be a connected, oriented surface. A bordered 3-manifold with boundary Σ is a pair (Y, ϕ) , where Y is a connected, oriented 3-manifold and $\phi : \partial Y \rightarrow \Sigma$ is an orientation preserving diffeomorphism.*

One can form a geometry whose points are isomorphism classes of bordered 3-manifolds with boundary Σ , and whose lines are surgery triangles. Denote this geometry by $X(\Sigma)$. Then Proposition 1 has the following generalization, due to Baldwin and Bloom [1].

Theorem 1. [1] *$X(\Sigma_g)$ is finitely generated. In fact, it has a generating set S_g of cardinality*

$$N(g) = \sum_{k=0}^g \binom{g}{k} C_k$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the k -th Catalan number.

In this paper we prove that some elements of S_g can be removed.

Theorem 2. $X(\Sigma_g)$ has a generating set $M_g \subset S_g$ of cardinality

$$n(g) = \frac{(2^g + 1)(2^{g-1} + 1)}{3}.$$

Definitions of M_g and S_g can be found in Section 2, and a proof of Theorem 2 can be found in Section 3. It turns out that the generating set S_g has two special redundancies, one in genus 5 and another in genus 6. These redundancies imply many more redundancies in higher genera, so the difference between $n(g)$ and $N(g)$ grows rapidly with g . We include for convenience a table of the numbers $n(g)$ and $N(g)$:

g	0	1	2	3	4	5	6	7
N(g)	1	2	5	15	51	188	731	2950
n(g)	1	2	5	15	51	187	715	2795

It turns out that there are no further redundancies - the generating set M_g is minimal. In Section 4 we will prove an even stronger result:

Theorem 3. Any generating set for $X(\Sigma_g)$ has cardinality at least $n(g)$.

To explain the concept behind our proof, it is useful to think about a sort of classifying space. Define a CW complex $B(\Sigma_g)$ by first taking an infinite wedge of circles, one for each bordered 3-manifold with boundary Σ_g , and then attaching a triangle for every surgery triangle $Y_\alpha, Y_\beta, Y_\gamma$. Note that there is a crucial ambiguity here in orienting the attaching maps. In section 4 we prove the following result, from which Theorem 3 follows as a corollary:

Proposition 2. There exists a choice of attaching maps so that $H_1(B(\Sigma_g); \mathbb{Z})$ is a free abelian group of rank $n(g)$.

Interestingly, our proof of Proposition 2 does not rely on the existence of M_g . Instead, it exploits a relationship between $X(\Sigma_g)$ and another geometry: the binary symplectic dual polar space, or $DSp(2g, 2)$. The points of this geometry are Lagrangian subspaces of \mathbb{F}_2^{2g} and the lines are triples of distinct Lagrangians whose intersection has dimension $g - 1$. In section 3 we construct an explicit surjective map $\mu : X(\Sigma_g) \rightarrow DSp(2g, 2)$ which takes surgery triangles to lines (or possibly tripled points). Under this map, any generating set for $X(\Sigma_g)$ maps to a generating set for $DSp(2g, 2)$.

It is a result of Brouwer that the “universal embedding dimension” of $DSp(2g, 2)$ is at least $n(g)$. The proof uses spectral graph theory and is sketched in [2]. Theorem 3 follows from Brouwer’s result, and the fact that the universal embedding dimension is a lower bound for the cardinality of any generating set for $DSp(2g, 2)$.

In fact, it was shown by Li [3], and independently Blokhuis and Brouwer [2], that the universal embedding dimension of $DSp(2g, 2)$ is exactly equal to $n(g)$. Our Theorem 2 implies a stronger result, that $DSp(2g, 2)$ has a spanning set of cardinality $n(g)$. This confirms a conjecture of Blokhuis and Brouwer.

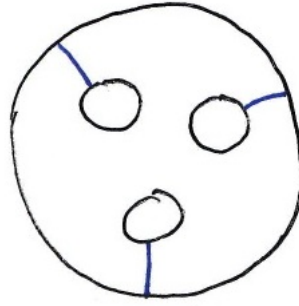
It should be noted that Li (and earlier McClurg [5]) proposed a spanning set for $DSp(2g, 2)$. However, their set is different from the image of M_g , so we have not verified that it generates.

The number $n(g)$ was initially suggested to us by some lengthy computer calculations. The author would like to thank Gabriel Gaster for his help in carrying out those calculations.

2. SPECIAL HANDLEBODIES

In this section we define a special set of bordered 3-manifolds. We will construct these manifolds by doing surgery on links in a thickened punctured disk.

Explicitly, let D be the unit disk in \mathbb{R}^2 , and let D_g^* be the punctured disk obtained by removing g smaller disks p_1, p_2, \dots, p_g from the interior of D . Thickening D_g^* yields a handlebody $H_g = D_g^* \times [-1, 1]$, whose boundary is a genus g surface Σ_g . We choose a collection of disjoint arcs $\alpha_1, \dots, \alpha_g \subset D_g^*$, which connect the punctures to the boundary of the disk as shown:



Let A_i denote the disk $\alpha_i \times [-1, 1]$, so that A_1, \dots, A_g are a set of compressing disks for the handlebody H_g .

Definition 2. We say that a framed link $L \subset H_g$ is minimal if it intersects each disk A_i at most once geometrically, and each of its components intersects at least one disk. We say that L is crossingless if it is contained in the slice $D_g^* \times \{0\}$.

It can be shown that any manifold obtained by surgery on a minimal crossingless link is homeomorphic to a handlebody, and that if two minimal crossingless links are not isotopic then the corresponding handlebodies are not homeomorphic. Strictly speaking we do not need either of these facts, but we will make use of them implicitly.

Definition 3. A bordered manifold H is said to be an almost-special handlebody if it is obtained by doing 0-surgery on each component of a minimal crossingless link $L \subset H_g$. We denote the set of almost-special handlebodies by S_g .

The set of almost-special handlebodies was previously considered by Baldwin and Bloom [1], although their description of these manifolds differs from ours. We now describe a smaller set $M_g \subset S_g$. The definition of M_g requires several concepts, which we now explain.

Using the distinguished arcs α_i , we can identify each puncture with a point on the boundary of D . This identification induces a cyclic ordering on the set of punctures. The definition of M_g requires us to choose a total ordering that is compatible with this cyclic ordering.

Equivalently, we can choose a “left-most” puncture p_1 , then list the remaining punctures p_2, p_3, \dots, p_g as they appear in clockwise order around the boundary.

Given a handlebody of genus g and a puncture p , we can form a new handlebody of genus $g - 1$ by filling p . When we fill a puncture, the total ordering on the punctures of D_g^* induces a total ordering on the punctures of D_{g-1}^* . This allows us to induct on the genus of a handlebody.

Definition 4. *We say that an almost-special handlebody H is reducible if the corresponding crossingless link L satisfies one of two conditions:*

- (1) *There is a puncture p which is not circled by any component of L .*
- (2) *There is a component of L which circles exactly one puncture.*

If H is not reducible, we say that it is irreducible. The reduction of a reducible handlebody is the irreducible handlebody obtained by first deleting each component of L that circles exactly one puncture, then filling every puncture that is not circled by any component.

Given the ordering on punctures, we can order the components of any minimal crossingless link L , by declaring that a component L_1 is to the left of another component L_2 if the leftmost puncture circled by L_1 is to the left of the leftmost puncture circled by L_2 . It is therefore sensible to talk about the leftmost component of a minimal crossingless link.

We now give an inductive definition of M_g .

Definition 5. *We say that an irreducible, almost-special handlebody is special if the corresponding crossingless link L satisfies each of the following three conditions:*

- (1) *The handlebody obtained by deleting the left-most component of L and filling every puncture it circles is special.*
- (2) *The punctures circled by the left-most component are consecutive with respect to the cyclic ordering (not necessarily with respect to the total ordering).*
- (3) *If the left-most component circles p_1, p_g , and p_{g-1} , then it circles every puncture.*

In general, we say that an almost-special handlebody is special if its reduction is special. We denote the set of special handlebodies by M_g .

Note that the empty link in a genus zero handlebody satisfies each condition vacuously, so even though the definition is inductive it does not require a base case.

Proposition 3. *For any $g \geq 1$, the number of irreducible special handlebodies is given by*

$$m(g) = \frac{2^{g-1} + (-1)^g}{3}.$$

Proof. It suffices to show that $m(g)$ satisfies the recursion

$$m(g) = m(g-1) + 2m(g-2),$$

together with the initial conditions $m(1) = 0$, $m(2) = 1$. The initial conditions are clear from the definitions. To prove the recursion, observe that any irreducible element of M_g is obtained in a unique way by one of three constructions:

- (1) Starting with an irreducible element of M_{g-1} , insert a puncture inside the left-most component, just to the right of the left-most puncture it contains.
- (2) Starting with an irreducible element of M_{g-2} , insert two new left-most punctures, and create a new component circling both of them.
- (3) Starting with an irreducible element of M_{g-2} , insert a new left-most puncture and a new right-most puncture, and create a new component circling both of them.

□

Be warned that the special case $m(0) = 1$ is not consistent with the formula above.

Proposition 4. *The number of special handlebodies is given by the formula*

$$n(g) = \frac{(2^g + 1)(2^{g-1} + 1)}{3}$$

Proof. A special handlebody is determined uniquely by a set uncircled punctures, a set of components that circle exactly one puncture, and an irreducible special handlebody. Therefore, the number of special handlebodies is equal to:

$$\sum_{k=0}^g \sum_{l=0}^{g-k} \binom{g}{k} \binom{g-k}{l} m(g-k-l)$$

where $m(h)$ is the number of irreducible handlebodies of genus h . This iterated summation can be carried out in an elementary (but tedious) manner, by repeatedly applying the binomial theorem. The result is $n(g)$. □

3. GENERATION

In this section, we prove that M_g generates $X(\Sigma_g)$. First we introduce some algebraic notation for the manifolds we are considering.

Observe that $X(\Sigma_g)$ has a simple binary operation, given by stacking thickened punctured disks. Given bordered manifolds Y_1 and Y_2 we denote the result of stacking Y_1 on top of Y_2 by $Y_1 Y_2$. We denote the result of 0-surgery on a minimal crossingless knot by the symbol $(i_1 i_2 \dots i_k)$, where $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ are the punctures circled by the knot, listed in increasing order.

A special handlebody can therefore be represented by a symbol like $(247)(56)(8)$. On the other hand, not every symbol like this corresponds to a special handlebody. For example, $(123)(13)$ is crossingless but not minimal, and $(13)(24)$ is minimal but not crossingless. Note that stacking is not commutative, so for example $(13)(24) \neq (24)(13)$.

Finally, we will also need to change the surgery coefficients on some links in our diagram. To represent p -framed surgery on a minimal crossingless knot, we will use a symbol like $(247)_p$.

Proposition 5. *To show that M_g generates $X(\Sigma_g)$, it suffices to show that the following two handlebodies lie in the span of M_g :*

- Handlebody $A =$ the genus 5 handlebody $(145)(23)$.

- *Handlebody B = the genus 6 handlebody $(14)(23)(56)$.*

Proof. By the result of Baldwin and Bloom, it suffices to show that every irreducible element of S_g lies in the span of M_g . Suppose that there is an irreducible H in S_g that is not generated by M_g . Since H is not special it violates one of the three conditions in the definition of special handlebody. If it violates the third condition then we can simplify it using the reduction of handlebody A . If it does not violate the third condition, but does violate the second condition, then we can simplify it using the reduction of handlebody B . If it does not violate the second and third conditions, but does violate the first, then we can forget about the leftmost component and proceed by induction on g . \square

We are therefore reduced to doing two explicit computations. There are two key results that make these computations possible.

Proposition 6. *The genus 3 handlebody (123) lies in the span of $(12)(23)$ and the reducible elements of M_3 .*

Proof. For any set of punctures X there is a surgery triangle relating X_p, X_{p+1} , and the empty diagram. Therefore, (123) lies in the span of the empty diagram and $(123)_1$, which in turn lies in the span of $(123)_1(13)$ and $(123)_1(13)_{-1}$. The former is equivalent by a handleslide to $(1)_1(13)$, which lies in the span of the reducible diagrams $(1)(13)$ and (13) . To simplify $(123)_1(13)_{-1}$ we can apply the famous lantern relation, in the form

$$(123)_1(13)_{-1} = (12)_1(23)_1(1)_{-1}(2)_{-1}(3)_{-1}$$

Modifying surgery coefficients on the right hand side of this equation shows that it lies in the span of $(12)(23)$ and reducibles, as desired. \square

We can easily modify the argument above to show that (123) lies in the span of $(23)(12)$ and reducibles. Thus Proposition 6 remains valid no matter how we permute the punctures.

Proposition 7. *The genus 4 handlebody $(12)(34)$ lies in the span of $(13)(24)$, $(14)(23)$, (1234) , and the reducible elements of M_4 .*

Proof. By the result of Baldwin and Bloom, $(13)(24)$ lies in the span of the special handlebodies. It is therefore enough to observe that there is a diffeomorphism ϕ of the standard genus 4 handlebody which takes the diagram of $(12)(34)$ to the diagram of $(13)(24)$, and whose inverse sends the diagram of any special handlebody to a diagram which lies in the span of $(13)(24)$, $(14)(23)$, (1234) , and the reducible elements of M_4 .

Such a diffeomorphism can be constructed as follows. Arrange the punctures 1, 2, 3, 4 so that they lie on the vertices of a square, with 1 in the upper right corner. Draw a vertical arc that separates punctures 1 and 4 from punctures 2 and 3. Flip over the part of H_g which lies to the right of the arc, thereby switching punctures 2 and 3. The result of applying this diffeomorphism to $(12)(34)$ is a two-component diagram whose components link punctures 1 and 3 and 2 and 4, respectively. This diagram is not crossingless, however we can make it crossingless by flipping over punctures 2 and 3 individually, in a direction opposite to the original flip. The combination of these three flips is a diffeomorphism $\phi : H_g \rightarrow H_g$ that takes $(12)(34)$ to $(13)(24)$.

Finally, one applies the inverse diffeomorphism ϕ^{-1} to all special handlebodies of genus 4. Using the result of Baldwin and Bloom for genera 2 and 3, one checks (tediously) that the resulting handlebodies all lie in the combined span of $(13)(24)$, $(14)(23)$, (1234) , and the reducible special handlebodies. \square

We are now ready to eliminate handlebodies A and B .

Proposition 8. *The handlebody $(145)(23)$ lies in the span of M_5 .*

Proof. We write $A \rightarrow X + Y + Z$ if the bordered manifold A lies in the span of X, Y, Z , and manifolds already known to be in the span of M_5 , like reducible handlebodies. The reduction is very complicated to write down in full, so we only show the important steps. The arrows below all follow from a combination of Propositions 6 and 7:

$$\begin{aligned}
(145)(23) &\rightarrow (15)(14)(23) \rightarrow (15)(12)(34) + (15)(13)(24) + (15)(1234) + (15)(234) \\
(15)(13)(24) &\rightarrow (13)(35)(24) \rightarrow (13)(23)(45) + (13)(25)(34) + (13)(2345) + (13)(245) \\
(13)(2345) &\rightarrow (13)(23)(345) + (13)(23)(45) \\
(13)(23)(345) &\rightarrow (12)(23)(345) \rightarrow (12345) + (123)(45) + (12)(345) + (13)(245) \\
(12)(25)(34) &\rightarrow (12)(35)(34) + (15)(23)(34) + (1235)(34) + (125)(34) \\
(1235)(34) &\rightarrow (12345) + (125)(34)
\end{aligned}$$

\square

Proposition 9. *The handlebody $(14)(23)(56)$ lies in the span of M_6 .*

Proof. Again, we show the important parts of the reduction:

$$\begin{aligned}
(14)(23)(56) &\rightarrow (13)(24)(56) + (12)(34)(56) + (1234)(56) \\
(13)(24)(56) &\rightarrow (13)(25)(46) + (13)(26)(45) + (13)(2456) \\
(13)(25)(46) &\rightarrow (12)(35)(46) + (15)(23)(46) + (1235)(46) \\
(13)(26)(45) &\rightarrow (12)(36)(45) + (16)(23)(45) + (1236)(45) \\
(13)(2456) &\rightarrow (12)(3456) + (1456)(23) + (123456) \\
(12)(35)(46) &\rightarrow (12)(34)(56) + (12)(36)(45) + (12)(3456) \\
(15)(23)(46) &= (23)(15)(46) \rightarrow (23)(16)(45) + (23)(14)(56) + (23)(1456) \\
(1235)(46) &\rightarrow (1234)(56) + (1236)(45) + (123456)
\end{aligned}$$

The only byproduct of the above reductions which does not lie in M_6 is $(23)(1456)$. Grouping together punctures 4 and 5 and applying Proposition 8 shows that this handlebody lies in the span of M_6 . \square

4. MINIMALITY

In this section we establish a lower bound for the cardinality of any generating set of $X(\Sigma_g)$. Instead of working directly with $X(\Sigma_g)$ we use its “universal embedding” (see [2] or [3], for example), or more precisely an integral lift of this embedding.

Definition 6. Let Σ be a connected oriented surface. We denote by $K(\Sigma)$ the free abelian group spanned by bordered 3-manifolds with boundary Σ , modulo the relations

$$Y_\alpha + Y_\beta + Y_\gamma = 0$$

for every surgery triangle $(Y_\alpha, Y_\beta, Y_\gamma)$.

There is a tautological map $X(\Sigma) \rightarrow K(\Sigma)$, and under this map any generating set for $X(\Sigma)$ maps to a spanning set for $K(\Sigma)$. The group $K(\Sigma)$ can be thought of as a sort of Grothendieck group of bordered 3-manifolds.

A subspace $L \subset H_1(\Sigma; \mathbb{F}_2)$ is said to be Lagrangian if it is isotropic with respect to the intersection pairing and has dimension g . If $S \subset H_1(\Sigma; \mathbb{F}_2)$ is an isotropic subspace of dimension $g-1$, then the quotient S^\perp/S is a 2-dimensional \mathbb{F}_2 vector space, so S is contained in exactly 3 Lagrangian subspaces. The relationship between these 3 Lagrangians is analogous to the relationship between the 3 manifolds involved in a surgery triangle, so we call such triples “isotropic triangles”.

Definition 7. Let Σ be a connected oriented surface. We denote by $L(\Sigma)$ the free abelian group spanned by Lagrangian subspaces $L \subset H_1(\Sigma; \mathbb{F}_2)$, modulo the relation

$$L_\alpha + L_\beta + L_\gamma = 0$$

for every isotropic triangle $(L_\alpha, L_\beta, L_\gamma)$. When $\Sigma = S^2$ we consider there to be a single Lagrangian subspace, the zero space, so that $L(S^2) = \mathbb{Z}$.

From now on all homology groups will have coefficients in \mathbb{F}_2 , so $H_1(\Sigma) = H_1(\Sigma; \mathbb{F}_2)$.

Proposition 10. If Y is a 3-manifold with (possibly disconnected) boundary Σ , then the kernel $L(Y)$ of the map $H_1(\Sigma) \rightarrow H_1(Y)$ is Lagrangian.

Proof. Let α and β be curves on Σ , bounding surfaces A and B in Y . If we perturb A and B to intersect transversely, then their intersection is a union of arcs, and these arcs together show that $\alpha \cap \beta$ has even cardinality. This shows that $L(Y)$ is isotropic.

Now consider the map

$$p : H_1(\Sigma)/L(Y) \rightarrow L(Y)^*$$

arising from the intersection pairing on $H_1(\Sigma)$. To show that $L(Y)$ has dimension exactly g , and is therefore Lagrangian, it suffices to show that p is injective.

Therefore, suppose that $\gamma \in H_1(\Sigma)$ is nonzero in $H_1(\Sigma)/L(Y)$. Pushing forward by the inclusion $j : \Sigma \rightarrow Y$, we obtain a class $j_*(\gamma)$ in $H_1(Y)$. This class is nonzero, because γ does not lie in $L(Y) = \ker j_*$. Therefore, by Poincaré duality, there is a class A in $H_2(Y, \Sigma)$ such that $j_*(\gamma) \cdot A$ is nonzero. Represent A by a surface in Y whose boundary lies on Σ , and let $\alpha = \partial A$. Then $\alpha \in L(Y)$ and $\gamma \cdot \alpha = j_*(\gamma) \cdot A$ is nonzero, so $p(\gamma)$ is nonzero as desired. \square

Proposition 11. If $Y_\alpha, Y_\beta, Y_\gamma$ form a surgery triangle, then one of two possibilities holds:

- (1) The corresponding Lagrangians $L_\alpha, L_\beta, L_\gamma$ form an isotropic triangle and we have

$$n_\alpha = n_\beta = n_\gamma$$

where n_α (for example) denotes the rank of $H_2(Y_\alpha)$ as an \mathbb{F}_2 vector space.

(2) *The corresponding Lagrangians are equal and we have*

$$(-2)^{n_\alpha} + (-2)^{n_\beta} + (-2)^{n_\gamma} = 0.$$

Proof. By assumption, the manifolds $Y_\alpha, Y_\beta, Y_\gamma$ are obtained by doing surgery on a knot K in some bordered manifold Y with boundary Σ_g . Denote by Z the manifold $Y \setminus \nu(K)$, whose boundary is $\Sigma_g \sqcup T^2$. Let α, β , and γ be the surgery curves on T^2 whose fillings give rise to Y_α, Y_β , and Y_γ .

Let \tilde{L} be the kernel of the inclusion map $H_1(T^2) \oplus H_1(\Sigma_g) \rightarrow H_1(Z)$. Then \tilde{L} is Lagrangian, so it has dimension $g + 1$. The kernel of the projection of \tilde{L} onto $H_1(T^2)$ is an isotropic subspace $S \subset H_1(\Sigma_g)$, which is contained in all three Lagrangians $L_\alpha, L_\beta, L_\gamma$. Since S is obtained by intersecting \tilde{L} with a codimension 2 subspace, it has dimension at least $g - 1$. Since it is isotropic it has dimension at most g . We therefore consider two separate cases.

If S has dimension $g - 1$, then the projection $\tilde{L} \rightarrow H_1(T^2)$ is surjective. Hence there are classes α', β' , and γ' in $H_1(\Sigma_g)$ that are homologous in Z to α, β , and γ . Let A, B , and C be surfaces witnessing these homologies. Then we have (for example)

$$\alpha' \cdot \beta' = A \cdot \beta' = A \cdot \beta = \alpha \cdot \beta = 1,$$

so the classes α', β' , and γ' are all distinct. Evidently these classes lie outside of S , and they bound in Y_α, Y_β , and Y_γ respectively, so together with S they span the Lagrangians L_α, L_β , and L_γ . Since these Lagrangians are all distinct, they form an isotropic triangle. Finally, note that a closed surface in Y_α (for example) cannot intersect the surgery curve in a homologically nontrivial way, because α does not bound in Z . Therefore, the Mayer-Vietoris sequence

$$H_2(T^2) \rightarrow H_2(Z) \oplus H_2(S^1) \rightarrow H_2(Y_\alpha) \rightarrow H_1(T^2)$$

shows that the map $H_2(Z)/H_2(T^2) \rightarrow H_2(Y_\alpha)$ is an isomorphism, and similarly for β and γ , so $n_\alpha = n_\beta = n_\gamma$.

If the dimension of S is g , then the image of the projection $\tilde{L} \rightarrow H_1(T^2)$ is 1-dimensional and $S = L_\alpha = L_\beta = L_\gamma$. Without loss of generality, the image of the projection is spanned by α . Without loss of generality, α is in the image of the projection, so there is some α' in $H_1(\Sigma)$ such that $(\alpha, \alpha') \in \tilde{L}$. By definition this means that α and α' are homologous in Z , so α' bounds in Y_α , and therefore $\alpha' \in S = L_\alpha$. We conclude that $\tilde{L} = \text{span}(\alpha) \oplus S$.

At any rate, we see that the kernel of $H_1(T^2) \rightarrow H_1(Z)$ is spanned by α , hence the Mayer-Vietoris sequence

$$H_2(Z) \oplus H_2(S^1) \rightarrow H_2(Y_{\alpha/\beta/\gamma}) \rightarrow H_1(T^2) \rightarrow H_1(Z) \oplus H_1(S^1)$$

shows that the rank of $H_2(Y_\alpha)$ is one greater than the common rank of $H_2(Y_\beta)$ and $H_2(Y_\gamma)$. The identity

$$(-2)^{n_\alpha} + (-2)^{n_\beta} + (-2)^{n_\gamma} = 0$$

then follows after a moment of thought about powers of 2. □

Motivated by Proposition 11, we define a map $\tilde{\mu} : X(\Sigma_g) \rightarrow L(\Sigma_g)$ by

$$\tilde{\mu}(Y) = (-2)^{n(Y)}[L(Y)]$$

where $n(Y)$ is the rank of $H_2(Y)$ as a vector space over \mathbb{F}_2 .

Proposition 12. *The map $\tilde{\mu}$ induces a surjective homomorphism $\mu : K(\Sigma_g) \rightarrow L(\Sigma_g)$.*

Proof. That $\tilde{\mu}$ induces a homomorphism follows directly from Proposition 11. That it is surjective follows from the fact that the homomorphism $\text{Mod}_g \rightarrow Sp_{2g}(\mathbb{F}_2)$ is surjective, and the fact that $Sp_{2g}(\mathbb{F}_2)$ acts transitively on Lagrangian subspaces. \square

In fact, μ is an isomorphism. Before proving this we make a simple observation:

Proposition 13. *If $Y \in X(\Sigma_g)$ and α is any simple closed curve on Σ_g whose homology class lies in $L(Y)$, then there exists $Y' \in X(\Sigma_g)$, equivalent to Y in $K(\Sigma_g)$, such that α bounds an embedded disk in Y' .*

Proof. Let $A \subset Y$ be a surface with boundary α . Without loss of generality, A is nonorientable, hence it is homeomorphic to a connected sum of real projective planes. Let e_1, \dots, e_k denote the exceptional curves in these projective planes. Removing a tubular neighborhood of each of these curves from Y , we obtain a manifold with k torus boundary components. With respect to the framings induced by A , the manifold Y is obtained by filling curves of slope $(1, 2)$ on each boundary torus. Let Y' be the result of filling the curves of slope $(1, 0)$ instead. Then the surgery triangles

$$Y_{(1,2)}(K) + Y_{(-1,-1)}(K) + Y_{(0,1)}(K) = 0$$

$$Y_{(1,0)}(K) + Y_{(-1,-1)}(K) + Y_{(0,1)}(K) = 0,$$

which are valid for any framed knot K in any 3-manifold Y , together show that $Y' = Y$ in $K(\Sigma_g)$. The effect of these surgeries on the surface A is to blow down all exceptional curves, the result being an embedded disk $A' \subset Y'$ with boundary α . \square

Theorem 4. *The map $\mu : K(\Sigma_g) \rightarrow L(\Sigma_g)$ is an isomorphism.*

Proof. First we treat the case $g = 0$. Since every 3-manifold can be reduced to S^3 by surgery triangles, we know that $K(S^2)$ is a cyclic group. Any generator of this cyclic group maps to a generator of $L(S^2) = \mathbb{Z}$, hence $K(S^2) = \mathbb{Z}$ and $\mu : K(S^2) \rightarrow L(S^2)$ is an isomorphism.

In general, any bordered manifold Y_0 with $L(Y_0) = L$ and $H_2(Y_0) = 0$ is equivalent in $K(\Sigma_g)$ to a “standard” example, namely the handlebody obtained by representing a standard basis of L by disjoint simple closed curves on Σ_g and attaching disks along these curves. To see why, suppose that $\alpha_1, \dots, \alpha_g$ are curves representing a basis for L . By Proposition 13, Y_0 is equivalent in $K(\Sigma_g)$ to another bordered manifold Y_1 in which α_1 bounds a disk. Examining the proof of Proposition 13, we see that $H_2(Y_0)$ and $H_2(Y_1)$ are isomorphic, so $H_2(Y_1) = 0$.

Continuing this process inductively produces a manifold Y_g , equivalent to Y_0 in $K(\Sigma_g)$, such that all the curves α_i bound disks in Y_g . This manifold Y_g is the connected sum of a standard handlebody H_g and a closed 3-manifold Z such that $H_2(Z) = 0$. Because $\mu : K(S^2) \rightarrow L(S^2)$ is an isomorphism and $H_2(Z) = 0$, a connected sum with Z is equivalent to a connected sum with S^3 , hence Y_g is equivalent to H_g in $K(\Sigma_g)$. Thus the original bordered manifold Y is equivalent to H_g , as claimed.

Any isotropic triangle can also be realized by a standard example, due to the transitive action of Mod_g on isotropic triangles. Hence there is an inverse map

$$\mu^{-1} : L(\Sigma_g) \rightarrow K(\Sigma_g)$$

defined by sending a Lagrangian L to any Y with $L(Y) = L$ and $H_2(Y) = 0$. Evidently we have $\mu^{-1}(\mu(Y)) = Y$ in $K(\Sigma_g)$, from which we conclude that μ is injective. \square

Any generating set for $X(\Sigma_g)$ maps to a spanning set for $K(\Sigma_g)$, or equivalently for $L(\Sigma_g)$. The group $L(\Sigma_g)$ has been computed by Blokhuis and Brouwer [2], who found that it is a free abelian group of rank

$$n(g) = \frac{(2^g + 1)(2^{g-1} + 1)}{3}.$$

Since any spanning set for $X(\Sigma_g)$ maps to a spanning set for $K(\Sigma_g)$, we obtain:

Corollary 1. *Any generating set for $X(\Sigma_g)$ has cardinality at least $n(g)$.*

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